Backpaper IInd semester 2016
B.Math.Hons.IIIrd year
Algebraic geometry — B.Sury

In what follows, K is an algebraically closed field.

Q 1.
In the ring $\mathbf{Z}[X]$, prove that $I = (4, X)$ is $(2, X)$-primary but that I is not the power of the maximal ideal $(2, X)$.

Q 2.
Let A be a commutative ring with unity. Show that if $\mathrm{Spec}(A)$ is not connected, then $A \cong A_1 \times A_2$ where the rings A_1, A_2 are both non-zero.

Q 3.
Let $\theta : A \to B$ be a K-algebra homomorphism of finitely generated K-algebras. By using the Noether normalization lemma or otherwise, deduce that $\theta^{-1}(\mathbf{m})$ is a maximal ideal of A for each maximal ideal $\mathbf{m}$ of B.

Q 4.
Let $V \subset \mathbf{A}^n(K)$ be an affine variety. Let $I(V)$ be its vanishing ideal and $A = K[X_1, \cdots, X_n]/I(V)$. For $f \in A$, consider $D(f) := \{x \in V : f(x) \neq 0\}$. Prove that each dense open subset U of V contains a dense set of the form $D(f)$.
Hint. Decompose $V = \cup_{i=1}^d V_i$, into irreducible components and observe that no V_i is contained in $V \setminus U$. Using the fact that an open subset is dense if and only if it meets every irreducible component, show you can get some $f \in I(V \setminus U)$ outside each $I(V_i)$ and that $D(f)$ is contained in U and is dense in V.

Q 5.
Let $\theta : \mathbf{A}^1_K \to V(Y^2 - X^3) \subset \mathbf{A}^2_K$ be the map

$$t \mapsto (t^2, t^3).$$

Show that θ is a bijective morphism and that the inverse of θ is not a morphism.